Final exam for Kwantumfysica 1 - 2010-2011
Thursday 21 April 2011, 13:00 - 16:00

READ THIS FIRST:

- Note that the lower half of this page lists some useful formulas and constants.
- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this exam has 3 questions, it continues on the backside of the papers!
- Start each question (number 1, 2, 3) on a new answer sheet.
- The exam is open book within limits. You are allowed to use the book by Liboff, the handout *Extra note on two-level systems and exchange degeneracy for identical particles*, and one A4 sheet with notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Standard Fourier transform pairs:

$$\Psi(x) = \begin{cases} \frac{1}{\sqrt{2b}}, & |x| \le b \\ 0, & |x| > b \end{cases} \quad \overset{Fourier}{\leftrightarrow} \quad \overline{\Psi}(k) = \sqrt{\frac{b}{\pi}} \frac{\sin kb}{kb}$$

$$\Psi(x) = \sqrt{\frac{b}{\pi}} \frac{\sin bx}{bx} \quad \overset{Fourier}{\leftrightarrow} \quad \overline{\Psi}(k) = \begin{cases} \frac{1}{\sqrt{2b}}, & |k| \le b \\ 0, & |k| > b \end{cases}$$

Standard integrals:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi}$$

Problem 1

Consider a one-dimensional system, with a single particle with mass $m = 10^{-20}$ kg at position x in the potential

$$V(x) = \tfrac{1}{2}\left(m\omega_0^2\right)x^2 .$$

Given the mass m, the constant ω_0 defines how steep the potential is. This system concerns a particle that is bound in a static potential, so it must have a discrete set of energy eigenstates $\chi_n(x)$ (or in Dirac notation, $|\chi_n\rangle$), where n is an index $n = 0,1,2,3\ldots$ for labeling these states.

a) Write down the Hamiltonian H of this system in x-representation. Write it out in an expression that uses the constants m and ω_0 where possible.

Assume that it is known that the ground state (lowest energy eigenstate) of this system is of the form

$$\Psi(x) = Ae^{-bx^2},$$

(in Dirac notation denoted as $|\Psi\rangle$) but that the values of A and b (real constants) are not known, and also the eigenvalue that belongs to this eigenstate is not known.

b) Draw a graph of $\Psi(x)$. For which value of A (in terms of constants b and others that you may need) is this state normalized?

In order to find the values for A and b for which the state $\Psi(x)$ represents the true ground state $\chi_0(x)$, you must use in this problem the *variational method.* For this case, this implies that $\langle \hat{H} \rangle$ is minimum with respect to the variation of the parameter b.

c) Say that the real (but still unknown to us) ground state energy of the system is E_0, with the corresponding eigenstate $|\chi_0\rangle$. Use Dirac notation to proof that for any state $|\Psi\rangle$ that we may consider, it will always obey $\dfrac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \geq E_0$.

Hint: Use that any trial state $|\Psi\rangle$ can always be written as a superposition of all the real energy eigenstates $|\chi_n\rangle$.

d) The results of **c)** shows that equality $\dfrac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = E_0$ holds only for the case $|\Psi\rangle = |\chi_0\rangle$.

Here $\dfrac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle}$ has a minimum value, so $|\chi_0\rangle$ and E_0 can be found by a procedure that minimizes the expression with respect to b. Obviously, this must be carried out in the x-representation. Use this approach to derive the values of b, A and E_0 in terms of m and ω_0.

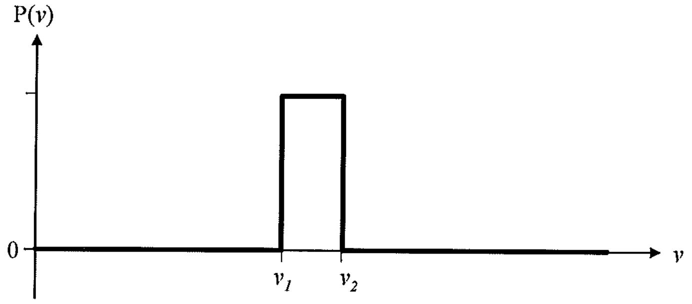
e) Calculate for the ground state that you found in **d)**, the expectation value for kinetic energy and the expectation value for potential energy. Explain the result of qualitatively in terms of the Heisenberg uncertainty relation.

Problem 2

An apparatus for an experiment shoots out electrons one by one, in one particular direction (the x-direction, the direction of the beam). After leaving the apparatus, the beam is passing an area where no significant forces act on the electron. Consequently, the electrons in the beam can be described as a free particle moving in one dimension.

a) Write down a Hamiltonian for one of the electrons in the beam. Explain your answer.

The research team aims at setting up the experiment in such a way that all the electrons leave the apparatus at the same speed, and that the quantum uncertainty in the speed of each electron is quite small. They aim at giving the electrons a velocity of 100 m/s. To check whether the setup works, they measure at some point in the beam (which they will define as $x = 0$) the velocity of a large number of electrons. They find a probability distribution $P(v)$ for the electrons' velocities v as in the figure below (uniform, with $v_1 = 198$ m/s and $v_2 = 202$ m/s).



b) They now remove this measurement apparatus from the beam, such that electrons passing $x = 0$ are not disturbed. Describe and sketch a normalized wavefunction $\overline{\Psi}(k)$ as a function of wavenumber k (in x-direction) for one of the electrons while it passes $x = 0$ (put in the sketch labels k_1 and k_2, related to v_1 and v_2). Use a wavefunction which agrees with the observed distribution $P(v)$, and show that it is normalized. Assume that the wavefunction can be chosen real and positive where it is not zero.

c) Calculate the wavefunction as a function of position x, that describes state of the electron which you already described as a function of k for answering question **b)**. Hint: write k_1 as k_c-Δk and k_2 as k_c+Δk (with $k_c = (k_1 + k_2)/2$, and $\Delta k = (k_2 - k_1)/2$).

d) Sketch and describe the probability density as a function of x, for the wavefunction you found in answer **c)**.

e) Estimate the uncertainty in the momentum and the position of the particle when it is near $x = 0$. Evaluate your answer.

f) They now put the measurement apparatus back into the beam. It measures the speed of the electrons without trapping the electron (that is, the electrons continue to fly along the beam). At some moment, an electron passes the measurement apparatus, and the velocity is measured with very high accuracy, with result v_m. Explain why one should now assume that the wavefunction as a function of k is very close to the form $\overline{\Psi}(k) = \delta(k - k_m)$. Calculate k_m for the case that the result is 199 m/s.

g) For the case of **f)**, consider the moment that the measurement was just finished, and the electron is still near $x = 0$. For this situation, calculate the wavefunction as a function of x that describes the state of the electron.

h) Sketch the probability density as a function of x, for the wavefunction you found in answer **g)**.

i) Evaluate the validity of the description of the state of the system in answers **f)-h)**.

j) One team members suggests to keep the velocity measurement apparatus in the beam, as it reduces the quantum uncertainty in the velocity of the electrons. Discuss whether this indeed is useful for meeting the goals that have been summarized above the figure. Discuss it for individual electrons, and for the ensemble of electrons.

Problem 3

A certain atom is in a state with its total orbital angular momentum vector $\boldsymbol{L}$ (described by the operator $\hat{L}$) defined by orbital quantum number $l = 1$. For the system in this state, the operator for the z-component of angular momentum is $\hat{L}_z$. It has three eigenvalues, $+\hbar$ (with corresponding eigenstate $|+_z\rangle$), $0\hbar$ (with eigenstate $|0_z\rangle$), and $-\hbar$ (with eigenstate $|-_z\rangle$). This operator can be represented as a matrix, and the ket-states as column vectors, using the basis spanned by $|+_z\rangle$, $|0_z\rangle$ and $|-_z\rangle$, according to

$$\hat{L}_z \leftrightarrow \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad |+_z\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \quad |-_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using this same basis for the representation, the operator and eigenstates for the system's x-component of angular momentum are given by

$$\hat{L}_x \leftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad |+_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad |0_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \text{ and } \quad |-_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}.$$

a) Use this information to *calculate* what the eigenvalues are that belong to $|+_x\rangle$, $|0_x\rangle$ and $|-_x\rangle$.

b) At some point the system is in the normalized state $|\Psi_1\rangle = \sqrt{\tfrac{1}{4}}\,|+_z\rangle + \sqrt{\tfrac{1}{4}}\,|0_z\rangle + i\,\sqrt{\tfrac{1}{2}}\,|-_z\rangle$. Calculate for this state the expectation value for angular momentum in z-direction and the expectation value for angular momentum in x-direction.

c) Calculate for this state $|\Psi_1\rangle$ the quantum uncertainty ΔL_z in the z-component of the system's angular momentum.

d) With the system still in this same state $|\Psi_1\rangle$, you are going to measure the x-component of the system's angular momentum. What are the possible measurement results? Calculate the probability for getting the measurement result with the highest value for angular momentum in x-direction.

e) Now the system is prepared in a different state (now superposition of eigenstates of $\hat{L}_x$), $|\Psi_2\rangle = \sqrt{\tfrac{1}{3}}\,|+_x\rangle - \sqrt{\tfrac{2}{3}}\,|-_x\rangle$. You are going to measure the z-component of the system's angular momentum. What is the probability to find the answer $+\hbar$?

f) Now the system is prepared in a different state (now again a superposition of eigenstates of $\hat{L}_z$), $|\Psi_3\rangle = \sqrt{\tfrac{1}{2}}\,|+_z\rangle + \sqrt{\tfrac{1}{2}}\,|0_z\rangle$, at time $t = 0$. Another change to the system is that one now applied an external magnetic field with magnitude B along the z-axis. The Hamiltonian of the system is now, $\hat{H} = \gamma B \hat{L}_z$, where γ is a constant that reflects how much the energy of angular momentum states shifts when applying the field. Calculate how the expectation value for angular momentum in x-direction depends on time.

Use Dirac notation and the operator $\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}}$.

Uitwerking Final Exam
Kwantumfysica 1 21 APRIL 2011

1/10

Problem 1

a) $\hat{H} = \hat{T} + \hat{V}$ (kinetic + ~~pt~~ potential energy)

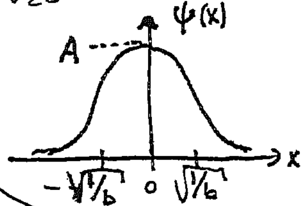
In x-representation, with constants used being $m, \omega_0 \Rightarrow$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2$$

b) $\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1$ when normalized $\Rightarrow$

$$\int_{-\infty}^{\infty} A^2 e^{-2bx^2} dx = 1 \Rightarrow A^2\int_{-\infty}^{\infty} e^{-(\sqrt{2b}x)^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}x) = 1 \Rightarrow$$

$$A^2 \frac{1}{\sqrt{2b}}\sqrt{\pi} = 1 \Rightarrow A = \left(\frac{2b}{\pi}\right)^{1/4}$$



c) Say $|\psi\rangle = \sum_{n=0}^{\infty} c_n |\chi_n\rangle \Rightarrow$

$$\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\sum_n E_n |c_n|^2}{\sum_n |c_n|^2} > \frac{\sum_n E_0 |c_n|^2}{\sum_n |c_n|^2}$$

$$= E_0 \frac{\sum_n |c_n|^2}{\sum_n |c_n|^2} = E_0 \quad \text{q.e.d.}$$

since all $E_n > E_0$ for $n > 0$

2/10

d) We need to minimize $\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}$ under variation of b.

Note that $\langle\psi|\psi\rangle$ always equals 1 if we always use $A = \left(\frac{2b}{\pi}\right)^{1/4}$ (from question b)), so we only need to minimize $\langle\psi|\hat{H}|\psi\rangle$ in that case, that is solve $\frac{d(\langle\psi|\hat{H}|\psi\rangle)}{db} = 0$.

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\hat{T}|\psi\rangle + \langle\psi|\hat{V}|\psi\rangle.$$

$$\langle\psi|\hat{V}|\psi\rangle = \sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} e^{-bx^2}\left(\frac{1}{2}m\omega_0^2 x^2\right)e^{-bx^2}dx$$

$$= \sqrt{\frac{2b}{\pi}}\,\frac{\frac{1}{2}m\omega_0^2}{2b}\int_{-\infty}^{\infty} 2bx^2\, e^{-2bx^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}x)$$

$$= \sqrt{\frac{2b}{\pi}}\,\frac{\frac{1}{2}m\omega_0^2}{2b}\,\frac{1}{\sqrt{2b}}\cdot\frac{1}{2}\sqrt{\pi} = \frac{m\omega_0^2}{8b}$$

$$\langle\psi|\hat{T}|\psi\rangle = \sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} e^{-bx^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)e^{-bx^2}dx$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} e^{-bx^2}\left(-2be^{-bx^2} + 4b^2x^2e^{-bx^2}\right)dx$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} -2b\,e^{-2bx^2} + 4b^2x^2e^{-2bx^2}\,dx$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\left(-2b\int_{-\infty}^{\infty} e^{-2bx^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}x) + 2b\int_{-\infty}^{\infty}(\sqrt{2b}x)^2 e^{-(\sqrt{2b}x)^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}x)\right)$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\left(\frac{-2b\sqrt{\pi}}{\sqrt{2b}} + \frac{2b\,\frac{1}{2}\sqrt{\pi}}{\sqrt{2b}}\right) = \frac{\hbar^2 b}{2m}$$

3/10

$$\Rightarrow \langle\psi|\hat{H}|\psi\rangle = \frac{m\omega_0^2}{8b} + \frac{\hbar^2 b}{2m} \Rightarrow$$

$$\frac{d\langle\psi|\hat{H}|\psi\rangle}{db} = \frac{d}{db}\left(\frac{m\omega_0^2}{8b} + \frac{\hbar^2 b}{2m}\right) = \frac{\hbar^2}{2m} - \frac{m\omega_0^2}{8b^2} = 0 \Rightarrow$$

$$\omega_0^2 = \frac{8\hbar^2 b^2}{2m} \Rightarrow b = \frac{m\omega_0}{2\hbar} \Rightarrow$$

$$\langle\psi|\hat{T}|\psi\rangle = \frac{\hbar^2 b}{2m} = \frac{1}{4}\hbar\omega_0$$

$$\langle\psi|\hat{V}|\psi\rangle = \frac{m\omega_0^2}{8b} = \frac{1}{4}\hbar\omega_0$$

$$\langle\psi|\hat{H}|\psi\rangle = E_0 = \frac{1}{2}\hbar\omega_0$$ (agrees indeed with harmonic oscillator ground state)

$$A = \left(\frac{2b}{\pi}\right)^{1/4} = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}$$

e) $\langle T\rangle = \frac{1}{4}\hbar\omega_0$, $\langle\hat{V}\rangle = \frac{1}{4}\hbar\omega_0$ (see d))

Heisenberg states $\Delta x \Delta p \geq \hbar/2$, so if the particle was truly at the bottom of the well, this would give $\langle\hat{V}\rangle = 0$ with $\Delta x = 0$.
Then, Δp must be very high, so $\langle\hat{T}\rangle$ very high and this high energy cost for $\langle\hat{T}\rangle$ makes that it is not the ground state. Instead, a trade off with both $\langle T\rangle$ and $\langle\hat{V}\rangle$ a bit more than zero gives a state with minimal energy.

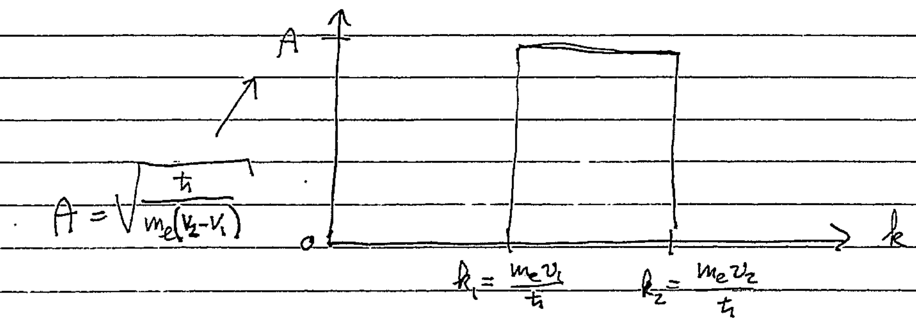
4/10

Problem 2

a) Free particle $\Rightarrow$ only kinetic energy, described with momentum operator $\hat{p}_x$ $\left(\text{or } -i\hbar\frac{\partial}{\partial x}\right)$

$$\hat{H} = \frac{\hat{p}_x^2}{2m_e} = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2}$$

b) $k = \frac{p_x}{\hbar} = \frac{m_e v}{\hbar}$



$$\int_{-\infty}^{\infty} |\bar{\psi}(k)|^2 dk = 1 \Rightarrow \int_{k_1}^{k_2} A^2 dk = 1 \Rightarrow$$

$$A^2(k_2 - k_1) = 1 \Rightarrow A = \sqrt{\frac{1}{(k_2-k_1)}} = \sqrt{\frac{\hbar}{m_e(v_2-v_1)}}$$

$$\bar{\psi}(k) = \begin{cases} \frac{1}{\sqrt{k_2-k_1}}, & k_1 < k < k_2 \\ 0 & \text{for all other } k \end{cases}$$

c) $$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{\psi}(k)\, e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{k_1}^{k_2} A e^{ikx} dk = \frac{A}{\sqrt{2\pi}}\left[\frac{1}{ix}e^{ikx}\right]_{k_1}^{k_2}$$

5/10

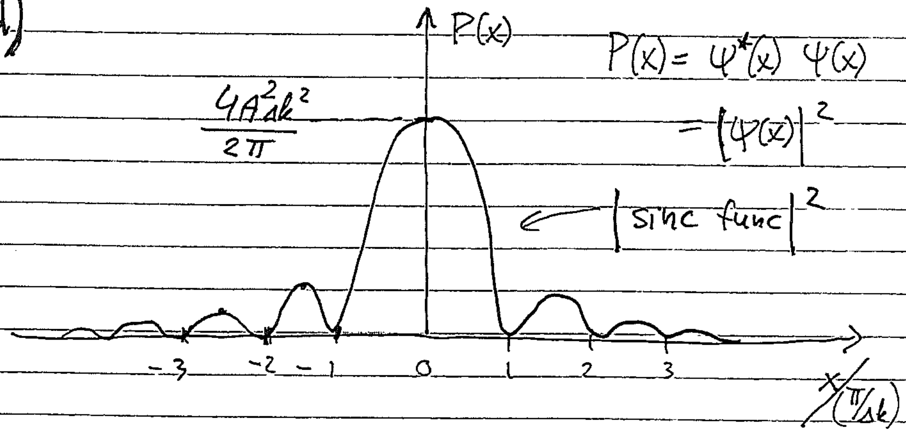
$$= -\frac{iA}{\sqrt{2\pi}\,x}\left(e^{ik_2x} - e^{ik_1x}\right)$$

$$= \frac{-iA}{\sqrt{2\pi}\,x}\left(e^{i(k_c+\Delta k)x} - e^{i(k_c-\Delta k)x}\right) = \frac{-iA}{\sqrt{2\pi}\,x}e^{ik_cx}\left(e^{i\Delta kx} - e^{-i\Delta kx}\right)$$

$$= -\frac{iA}{\sqrt{2\pi}\,x}e^{ik_cx}\left(2i\sin(\Delta k x)\right) \Rightarrow$$

$$\psi(x) = \underbrace{\left(\frac{2A\Delta k}{\sqrt{2\pi}}\right)}_{\text{Amplitude}}\underbrace{\left(e^{ik_cx}\right)}_{\text{phase factor}}\cdot\underbrace{\frac{\sin(\Delta k x)}{\Delta k x}}_{\text{sinc function}}$$

d)



$$P(x) = \psi^*(x)\,\psi(x) = |\psi(x)|^2$$

$\leftarrow |\text{sinc func}|^2$

e) From the sketch of question b)

$\Delta p_x \approx m_e(v_2 - v_1)/\hbar$

From the sketch of question d)

$\Delta x \approx \frac{2\pi}{\Delta k} = \frac{4\pi\hbar}{m_e(v_2-v_1)}$

$\Delta p_x \cdot \Delta x = \frac{4\pi\hbar}{m_e}\cdot\frac{m_e}{2} \approx 2\pi\hbar$

$\Rightarrow$ close to the Heisenberg limit

6/10

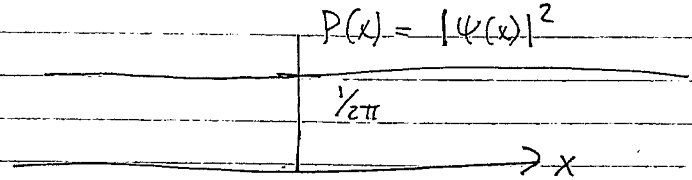
f) Measurement of v collapses the state of an electron at a state with a very specific value of speed v_m (when the measurement result is v_m) $\Rightarrow$
As a function of k the state is close to a Dirac delta function centered at k_m $\Rightarrow$ $\bar{\varphi}(k) = \delta(k - k_m)$

$$k_m = \frac{m_e v_m}{\hbar} = \frac{9.1\cdot 10^{-31}\,kg \cdot 199\,m/s}{1.055\cdot 10^{-34}\,Js} = 1.7\cdot 10^{6}\,m^{-1}$$

g) $$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{\varphi}(k)e^{ikx}dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(k-k_m)e^{ikx}dk \Rightarrow$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}}e^{ik_mx} \Rightarrow \text{a plane wave}$$

h)



$$P(x) = |\psi(x)|^2 = \psi^*(x)\psi(x) = \frac{1}{2\pi}$$

i) The uncertainty in p_x (and thereby k) is very small, close to zero.
The uncertainty in position (called Δx) the becomes very large ($\sim$ infinite)
This can of course not happen in a real experimental setup, but indicates what happens: Δp_x very small and Δx very large.

7/10

j) For an individual electron, the quantum uncertainty in the velocity is indeed very small.
However, $\langle v \rangle$ will not always be exactly 200 m/s.
Instead, it will have some value in the range $v_1 \cdots v_2$.
This value is prepared in a probabilistic manner (by measurement) but one does know the value $\langle v \rangle = v_m$ from the measurement result.
It will be different for each electron.
For an ensemble of electrons, there is no change in the spread of velocities. The probability distribution $P(v)$ will still be as in the figure in the question.

Problem 3

8/10

a) The eigen values can be calculated with the eigen value equations for $\hat{L}_x$

For $|+_x\rangle$, $\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 1\\ \frac{1}{\sqrt{2}}\end{pmatrix} = +\hbar\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix} \Rightarrow$ eigen value is $+\hbar$

For $|0_x\rangle$, $\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0\\0\\0\end{pmatrix} = 0\hbar\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix} \Rightarrow$ eigen value is $0\hbar$ ($\hbar$ for unit)

For $|-_x\rangle$, $\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ 1\\ -\frac{1}{\sqrt{2}}\end{pmatrix} = -\hbar\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix} \Rightarrow$ eigen value is $-\hbar$

b) $\langle \hat{L}_z \rangle = \langle \psi_1 | \hat{L}_z | \psi_1 \rangle$

$$= \left(\sqrt{\tfrac{1}{4}} \quad \sqrt{\tfrac{1}{4}} \quad -i\sqrt{\tfrac{1}{2}}\right)\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\sqrt{1/4}\\ \sqrt{1/4}\\ +i\sqrt{\frac{1}{2}}\end{pmatrix}$$

$$= \left(\sqrt{\tfrac{1}{4}} \quad \sqrt{\tfrac{1}{4}} \quad -i\sqrt{1/2}\right)\hbar\begin{pmatrix}\sqrt{1/4}\\ 0\\ -i\sqrt{\frac{1}{2}}\end{pmatrix} = \left(\frac{1}{4} + 0 - \frac{1}{2}\right)\hbar = -\frac{1}{4}\hbar$$

$\langle L_x \rangle = \langle \psi_1 | L_x | \psi_1 \rangle$

$$= \left(\sqrt{\tfrac{1}{4}} \quad \sqrt{\tfrac{1}{4}} \quad -i\sqrt{\tfrac{1}{2}}\right)\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}\sqrt{1/4}\\ \sqrt{1/4}\\ +i\sqrt{1/2}\end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}}\left(\sqrt{\tfrac{1}{4}} \quad \sqrt{\tfrac{1}{4}} \quad -i\sqrt{\tfrac{1}{2}}\right)\begin{pmatrix}\sqrt{1/4}\\ \sqrt{1/4}+i\sqrt{1/2}\\ \sqrt{1/4}\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\left(\frac{1}{4} + \frac{1}{4} + i\sqrt{\frac{1}{8}} - i\sqrt{\frac{1}{8}}\right) = \frac{\hbar}{2\sqrt{2}}$$

9/10

c) $\Delta L_z = \sqrt{\langle L_z^2\rangle - \langle L_z\rangle^2}$

So we first calculate the matrix representation of L_z^2

$$(L_z)^2 \leftrightarrow \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\hbar^2 = \hbar^2\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

Then, $\langle L_z^2\rangle = \langle\psi_1|L_z^2|\psi_1\rangle$

$$= \left(\sqrt{\tfrac14}\ \ \sqrt{\tfrac14}\ \ -i\sqrt{\tfrac12}\right)\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}\sqrt{1/4}\\\sqrt{1/4}\\i\sqrt{\frac12}\end{pmatrix}\hbar^2 = \left(\sqrt{\tfrac14}\ \ \sqrt{\tfrac14}\ \ -i\sqrt{\tfrac12}\right)\begin{pmatrix}\sqrt{1/4}\\0\\+i\sqrt{\frac12}\end{pmatrix}\hbar^2$$

$$= \frac14\hbar^2 + \frac12\hbar^2 = \frac34\hbar^2$$

Use from b) $\langle L_z\rangle = -\frac14\hbar \Rightarrow \Delta L_z = \sqrt{\frac34\hbar^2 - \frac1{16}\hbar^2} = \sqrt{\frac{11}{16}}\hbar$

d) You measure L_x for $l=1$. From problem a) or general theory for L_x for $l=1$, the possible results are the eigen values of $\hat{L}_x$, which are $\underline{+\hbar\quad, 0\hbar \quad \text{and} -\hbar}$

$$P_{+\hbar,x} = |\langle +_x|\psi_1\rangle|^2 = \left|\left(\tfrac12\ \ \tfrac{1}{\sqrt2}\ \ \tfrac12\right)\begin{pmatrix}\sqrt{1/4}\\\sqrt{1/4}\\i\sqrt{1/2}\end{pmatrix}\right|^2$$

$$= \left|\left(\frac14 + \frac{1}{2\sqrt2} + i\frac{1}{2\sqrt2}\right)\right|^2 = \frac{5}{16} + \frac18\sqrt2$$

≈ 0.49

10/10

e) $P_{+\hbar,z} = |\langle +_z|\psi_2\rangle|^2$

$$= \left|(1\,0\,0)\left(\sqrt{\tfrac13}\begin{pmatrix}\frac12\\1/\sqrt2\\1/2\end{pmatrix} - \sqrt{\tfrac23}\begin{pmatrix}1/2\\-1/\sqrt2\\1/2\end{pmatrix}\right)\right|^2$$

$$= \left|(1\,0\,0)\begin{pmatrix}\frac12\sqrt{\frac13} - \frac12\sqrt{\frac23}\\ \frac{1}{\sqrt6} + \sqrt{2/6}\\ -\frac12\sqrt{\frac13} - \frac12\sqrt{2/3}\end{pmatrix}\right|^2 = \frac14\left(\sqrt{\tfrac13} - \sqrt{\tfrac23}\right)^2$$

$$= \frac{(\sqrt2 - 1)^2}{12} \approx 0.015$$

f) $\langle L_x(t)\rangle = \langle\psi(t)|\hat{L}_x|\psi(t)\rangle$ with $|\psi(t)\rangle = \hat{U}|\psi_0\rangle = \hat{U}|\psi_3\rangle$

$$\hat{H} \leftrightarrow \begin{pmatrix}E_+&0&0\\0&0&0\\0&0&E_-\end{pmatrix} = \hbar\begin{pmatrix}\omega_+&0&0\\0&0&0\\0&0&-\omega_-\end{pmatrix} \text{ with } \begin{cases}E_+ = +\gamma B\hbar\\ E_0 = 0\\ E_- = -\gamma B\hbar\\ \omega_+ = +\gamma B\\ \omega_0 = 0\\ \omega_- = -\gamma B\end{cases}$$

$$\Rightarrow |\psi(t)\rangle = \frac{e^{i\omega_+ t}}{\sqrt2}|+_z\rangle + e\frac{-i\omega_0 t}{\sqrt2}|0_z\rangle$$

$$\Rightarrow \langle L_x(t)\rangle = \left(e^{+i\omega_+ t},\ e^{+i\omega_0 t},\ 0\right)\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}e^{-i\omega_+ t}\\e^{-i\omega_0 t}\\0\end{pmatrix}\cdot\frac{\hbar}{\sqrt2}\cdot\frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2}$$

$$= \frac{\hbar}{2\sqrt2}\left(e^{+i(\omega_+-\omega_0)t} + e^{-i(\omega_+-\omega_0)t}\right)$$

$$= \frac{\hbar}{\sqrt2}\cos\left((\omega_+-\omega_0)t\right) = \frac{\hbar}{\sqrt2}\cos(\gamma B t)$$